Applications of Polynomial Algebras to 2-Dimensional Deformed Oscillators

Ci Song,* Fu-Lin Zhang,† and Jing-Ling Chen‡
Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

(Dated: October 20, 2009)

The polynomial algebra is a deformed su(2) algebra. Here, we use polynomial algebra as a method to solve a series of deformed oscillators. Meanwhile, we find a series of physics systems corresponding with polynomial algebra with different maximal order.

PACS numbers: 02.20.-a; 03.65.Fd; 03.65.Ge; 03.65.-w

I. INTRODUCTION

The idea of using physical systems symmetries to study degenerate energy levels has been adopted since the early days of quantum mechanics. So ladder operators which connect all the eigen-states with a given energy lead a good method to solve this problem. For linear systems, such as Hydrogen atom and isotropic harmonic oscillator, Lie algebra can work out these problems well. Generally, the N-dimensional hydrogen atom has the so$(N+1)$ and the oscillator has the su(N) symmetry.

Afterwards, Higgs [1] and Leemon [2] introduced a generalization of the hydrogen atom and isotropic harmonic oscillator in a space with constant curvature. In Higgs' literature [1], he constructed a new algebra isomoriphic to so(3) and su(2) to describe the symmetry of hydrogen atom and isotropic harmonic oscillator on 2-dimensional sphere and this new algebra is called Higgs algebra which is also used in two-body Calogero-Sutherland model [3] and Karassiov-Klimov model [4]. Then, additional examples, like the Fokas-Lagerstrom potential [5], the Smorodinsky-Winternitz potential [6], and the Holt potential [7], were finally solved by Dennis Bonatsos et al [8] in the method of ladder operators.

The polynomial algebra [9] is a deformation of normal angular algebra su(2), which owns three generators J_0, J_+ and J_-. However, the commutative relation of J_+ and J_- appears the polynomial of J_0. su(2) and Higgs algebra are both special cases of polynomial algebra. It can be represented as $\mathfrak{J}^{(\Omega)}$, where Ω is a positive integer which expresses the highest power of the polynomial. The generators J_0, J_+ and J_- of $\mathfrak{J}^{(\Omega)}$ satisfy

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = P(J_0), \tag{1}$$

and its Casimir operator can be written as

$$C^{(\Omega)} = \{J_+, J_-\} + \sum_{i=0}^{\Omega+1} \alpha_i J_0^i. \tag{2}$$

Here, in this paper, we expand the Fokas-Lagerstrom potential and the Holt potential to the oscillator's frequency satisfying $\omega_1 : \omega_2 = l_1 : l_2$ which is integer ratio. Thus, with this result, we can easily get Bonatsos' result.

II. POLYNOMIAL ALGEBRA METHOD

For a 2-dimensional physical system exhibiting dynamical symmetry, we can find a set of operators J_0, J_+ and J_- which communicate with the Hamiltonian of system and satisfy (1) as ladder operators.

Firstly, we assume the dimension of representation of this system is finite. So there must be an upper bound $|\overline{m}\rangle$ and a lower bound $|\underline{m}\rangle$ in each degenerate energy level.

*Email:cisong@mail.nankai.edu.cn
†Email:flzhang@mail.nankai.edu.cn
‡Email:chenjl@nankai.edu.cn

Meanwhile, because of (1), it is easy to see $[J_+J_-, J_0] = [J_-J_+, J_0] = 0$. So we know J_+J_- and J_-J_+ must be the function of J_0 and H

$$J_+J_- = \phi(J_0, H), \quad J_-J_+ = \phi(J_0 + 1, H) \tag{3}$$

Thus, we use them to act on $|\overline{\mathrm{m}}\rangle$ and $|\underline{\mathrm{m}}\rangle$ respectively. We get equations

$$J_+J_- |\underline{\mathrm{m}}\rangle = \phi(\underline{\mathrm{m}}, E) |\underline{\mathrm{m}}\rangle = 0, \quad J_-J_+ |\overline{\mathrm{m}}\rangle = \phi(\overline{\mathrm{m}} + 1, E) |\overline{\mathrm{m}}\rangle = 0. \tag{4}$$

In both equations, we can omit energy E and require $\overline{\mathrm{m}} - \underline{\mathrm{m}} = n$ is integer. Finally, we could omit part of results which cause energy E goes to negative infinity when n goes to positive infinity. Then, we finally get the energy level and degenerate degree.

III. USING POLYNOMIAL ALGEBRA IN 2-DIMENSIONAL DEFORMED OSCILLATORS

A. 2-Dimension isotropic harmonic oscillator

Firstly, we use 2-D isotropic harmonic oscillator as an example. Its Hamiltonian can be written as

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2} m\omega^2 (x_1^2 + x_2^2). \tag{5}$$

If we write operators

$$a_i = \sqrt{\frac{m\omega_i}{2\hbar}} x_i + i \frac{p_i}{\sqrt{2m\omega_i\hbar}}, \quad a_i^\dagger = \sqrt{\frac{m\omega_i}{2\hbar}} x_i - i \frac{p_i}{\sqrt{2m\omega_i\hbar}} \quad (i = 1, 2) \tag{6}$$

and

$$N_i = a_i^\dagger a_i = \frac{1}{\hbar\omega_i} \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega_i^2 x_i^2\right) - \frac{1}{2} \quad (i = 1, 2), \tag{7}$$

We can rewrite the Hamiltonian as the following form

$$H = \hbar\omega(N_1 + N_2 + 1) \tag{8}$$

1. Normal method

   Usually it is solved by second order tensors [10].

   $$S_0 = \frac{1}{2}(N_1 - N_2), \quad S_+ = a_1^\dagger a_2, \quad S_- = a_1 a_2^\dagger. \tag{9}$$

   Their Casimir operator C can be write as

   $$C = \frac{1}{2}\{S_+, S_-\} + S_0^2, \tag{10}$$

   and energy level can be solved as

   $$E_n = \hbar\omega(n + 1), \quad n = n_1 + n_2, \quad n_1, n_2 = 0, 1, 2, \cdots \tag{11}$$

   where $n = 0, 1, 2, \cdots$, and there are $n + 1$ degenerate eigenstates for each energy level E_n.

2. Polynomial algebra method

   If we use new operators

   $$J_0 = \frac{1}{4}(N_1 - N_2), \quad J_+ = (a_1^\dagger)^2 (a_2)^2, \quad J_- = (a_1)^2 (a_2^\dagger)^2. \tag{12}$$

   We could find their communicative relations

   $$[J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-, \tag{13}$$

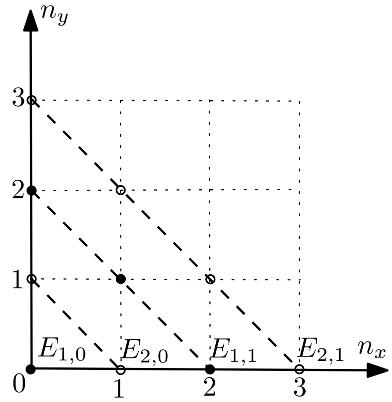

FIG. 1: As shown in the figure, the solid point is represented the energy state described by Eq.(16a) and the hollow point is represented the energy state described by Eq.(16b). The dashed line shows the degenerate eigenstates.

and

$$[J_+, J_-] = 4\left(\frac{H^2}{\hbar^2\omega^2} - 3\right) J_0 - 64J_0^3. \tag{14}$$

From the communicative relation, it is obvious that $J_+,\ J_-$ and J_0 satisfies Higgs algebra relation[1], which the maximal order of J_0 in $[J_+, J_-]$ is 3. Meanwhile, we can get their Casimir operator

$$C = \frac{1}{8}\left(\frac{H}{\hbar\omega}\right)^4 - \frac{5}{4}\left(\frac{H}{\hbar\omega}\right)^2 + \frac{9}{8}, \tag{15}$$

and energy level

$$E_{1,n} = \hbar\omega(2n+1) \tag{16a}$$
$$E_{2,n} = \hbar\omega(2n+2) \tag{16b}$$

where $n = 0, 1, 2, \cdots$, and there are $2n + i$ degenerate eigenstates for each energy level E_{in}, $i = 1, 2$. As shown in Fig1, the solid point is represented the energy state described by Eq.(16a) and the hollow point is represented the energy state described by Eq.(16b).

Comparing above two methods, we can see the polynomial algebra can also give all the energy level for the system. More exciting, it could be used for other deformed oscillator or non-linear potential.

B. 2-Dimensional anisotropic harmonic oscillator

The Hamiltonian of 2-D anisotropic harmonic oscillator can be written as

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}m(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) = (N_1 + \frac{1}{2})\hbar\omega_1 + (N_2 + \frac{1}{2})\hbar\omega_2. \tag{17}$$

If $\omega_1 : \omega_2 = l_1 : l_2$ is integer ratio, we can write $\omega_1 = l_1\omega_0, \omega_2 = l_2\omega_0$ and construct new operators

$$J_0 = \frac{1}{2}\left(\frac{N_1}{l_2} - \frac{N_2}{l_1}\right), \quad J_+ = (a_1^\dagger)^{l_2}(a_2)^{l_1}, \quad J_- = (a_1)^{l_2}(a_2^\dagger)^{l_1}. \tag{18}$$

We could find their communicative relations

$$[J_0, J_+] = J_+, \quad [J_0, J_-] = -J_- \tag{19}$$

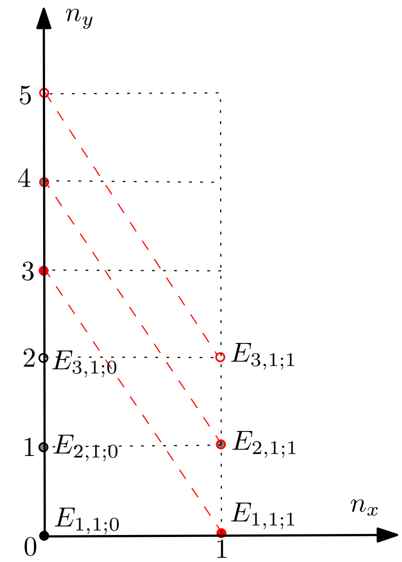

FIG. 2: As shown in the figure(color online), the solid point is represented the energy state described by Eq.(25a); the grey point is represented the energy state described by Eq.(25b); the hollow point is represented the energy state described by Eq.(25c). The red dashed line shows the degenerate eigenstates when $n = 1$.

and

$$
\begin{aligned}
[J_+, J_-] \;=\; & \prod_{i=1}^{l_2}\left(\frac{2H-\hbar\omega_2}{4\hbar\omega_1}+l_2J_0-i+\frac{3}{4}\right)\cdot\prod_{j=1}^{l_1}\left(\frac{2H-\hbar\omega_1}{4\hbar\omega_2}-l_1J_0+j-\frac{1}{4}\right) \\
& -\prod_{i=1}^{l_2}\left(\frac{2H-\hbar\omega_2}{4\hbar\omega_1}+l_2J_0+i-\frac{1}{4}\right)\cdot\prod_{j=1}^{l_1}\left(\frac{2H-\hbar\omega_1}{4\hbar\omega_2}-l_1J_0-j+\frac{3}{4}\right).
\end{aligned}
\tag{20}
$$

which the maximal order of J_0 in $[J_+, J_-]$ is $l_1 + l_2 - 1$ corresponding to the polynomial algebras with $l_1 + l_2 - 1$ order. We can solve their Casimir operator

$$
\begin{aligned}
C \;=\; & \prod_{i=1}^{l_2}\left(\frac{2H-\hbar\omega_2}{4\hbar\omega_1}-i+\frac{3}{4}\right)\cdot\prod_{j=1}^{l_1}\left(\frac{2H-\hbar\omega_1}{4\hbar\omega_2}+j-\frac{1}{4}\right) \\
& +\prod_{i=1}^{l_2}\left(\frac{2H-\hbar\omega_2}{4\hbar\omega_1}+i-\frac{1}{4}\right)\cdot\prod_{j=1}^{l_1}\left(\frac{2H-\hbar\omega_1}{4\hbar\omega_2}-j+\frac{3}{4}\right).
\end{aligned}
\tag{21}
$$

and energy level

$$
E_{i,j;n} = \hbar\omega_1(i-\frac{1}{2}) + \hbar\omega_2(j-\frac{1}{2}) + \hbar\frac{\omega_1\omega_2}{\omega_0}n \tag{22}
$$

where $n = 0, 1, 2, \cdots$, and there are $n+1$ degenerate eigenstates for each energy level $E_{i,j;n}$, $i = 1, \cdots, l_2$, $j = 1, \cdots, l_1$, which different i and j numbers show different formulae for the energy levels.

When $l_1 : l_2 = 3 : 1$, it could be viewed as Fokas-Lagerstorm potential, which is taken as an example here. For Fokas-Lagerstorm potential, we can calculate its communicative relation as

$$
\begin{aligned}
[J_+, J_-] \;=\; & \frac{1}{64\hbar^3\omega_1\omega_2^3}\left(-8H^3(\omega_1-9\omega_2)+12\hbar H^2(\omega_1^2-4\omega_1\omega_2+3\omega_2^2)-\right. \\
& \left.2\hbar^2H(3\omega_1^3+3\omega_1^2\omega_2+77\omega_1\omega_2^2-51\omega_2^3)+\hbar^3(\omega_1^4+6\omega_1^3\omega_2+68\omega_1^2\omega_2^2-6\omega_1\omega_2^3-69\omega_2^4)\right) \\
& +\frac{3\left(12H^2(\omega_1-3\omega_2)-12\hbar H\omega_1(\omega_1-\omega_2)+\hbar^2(3\omega_1^3+3\omega_1^2\omega_2+41\omega_1\omega_2^2+9\omega_2^3)\right)}{8\hbar^2\omega_1\omega_2^2}J_0 \\
& +\frac{81\left(-2H\omega_1+\hbar\omega_1^2+2H\omega_2-\hbar\omega_2^2\right)}{4\hbar\omega_1\omega_2}J_0^2+108J_0^3
\end{aligned}
\tag{23}
$$

which the maximal order of J_0 in $[J_+, J_-]$ is 3 corresponding to the polynomial algebras with 3 order. The Casimir operator can be expressed as

$$\begin{aligned} C &= -\frac{1}{128\hbar^4\omega_1\omega_2^3}\big(-16H^4 + 16\hbar H^3(\omega_1-\omega_2) + 16\hbar^2 H^2(9\omega_1 - 23\omega_2)\omega_2 \\ &\quad -4\hbar^3 H(\omega_1^3 + 33\omega_1^2\omega_2 - 33\omega_1\omega_2^2 - \omega_2^3) + \hbar^4(\omega_1^4 + 32\omega_1^3\omega_2 + 26\omega_1^2\omega_2^2 + 88\omega_1\omega_2^3 + 93\omega_2^4)\big) \end{aligned} \tag{24}$$

the energy level calculated by polynomial algebra can array as follows

$$E_{1,1;n} = \frac{1}{2}\hbar\omega_1 + \frac{1}{2}\hbar\omega_2 + \hbar\frac{\omega_1\omega_2}{\omega_0}n, \tag{25a}$$

$$E_{2,1;n} = \frac{3}{2}\hbar\omega_1 + \frac{1}{2}\hbar\omega_2\hbar\frac{\omega_1\omega_2}{\omega_0}n, \tag{25b}$$

$$E_{3,1;n} = \frac{5}{2}\hbar\omega_1 + \frac{1}{2}\hbar\omega_2 + \hbar\frac{\omega_1\omega_2}{\omega_0}n. \tag{25c}$$

For $l_1 = 3$ and $l_2 = 1$, it is clearly that the energy levels have three formula forms, which the number of energy level formulae equals to $l_1 \times l_2$. As shown in Fig2, we have known that the number of degenerate eigenstates equals to $n+1$ for each energy level formula.

C. 2-Dimensional anisotropic harmonic oscillator with Smorodinsky-Winternitz potential

The Hamiltonian of 2-Dimensional anisotropic harmonic oscillator with Smorodinsky-Winternitz potential system can be written as

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}m\omega_1^2 x_1^2 + \frac{1}{2}m\omega_2^2 x_2^2 + \frac{\kappa}{2x_2^2} \tag{26}$$

where $V_I = \frac{\kappa}{2x_2^2}$ is hard to deal with. We can construct operators

$$A_1 = a_1^2, \tag{27a}$$

$$A_1^\dagger = (a_1^\dagger)^2 \tag{27b}$$

$$A_2 = a_2^2 - \frac{V_I}{\hbar\omega_2}, \tag{27c}$$

$$A_2^\dagger = (a_2^\dagger)^2 - \frac{V_I}{\hbar\omega_2}, \tag{27d}$$

and rewrite the Hamiltonian as

$$H = H_1 + H_2, \quad H_1 = (N_1 + \frac{1}{2})\hbar\omega_1, \quad H_2 = (N_2 + \frac{1}{2})\hbar\omega_2 + V_I \tag{28}$$

They satisfy communicative relations as

$$[H_i, A_j] = -2\hbar\omega_i A_i\delta_{ij}, \quad \left[H_i, A_j^\dagger\right] = 2\hbar\omega_i A_i^\dagger\delta_{ij}, \quad \left[A_i, A_j^\dagger\right] = \frac{4}{\hbar\omega_i}H_i\delta_{ij}. \tag{29}$$

So, for total Hamiltonian (26), we have

$$H(A_1)^{l_2}(A_2^\dagger)^{l_1} = (A_1)^{l_2}(A_2^\dagger)^{l_1}(H + 2l_1\hbar\omega_2 - 2l_2\hbar\omega_1) \tag{30}$$

which means that, if $\omega_1 = l_1\omega_0,\ \omega_2 = l_2\omega_0$ is integer ratio, we have $[H, (A_1)^{l_2}(A_2^\dagger)^{l_1}] = 0$. So we can construct the ladder operators

$$J_0 = \frac{1}{2(l_1+l_2)\hbar}\left(\frac{H_1}{\omega_1} - \frac{H_2}{\omega_2}\right), \quad J_+ = (A_1^\dagger)^{l_2}A_2^{l_1}, \quad J_- = A_1^{l_2}(A_2^\dagger)^{l_1} \tag{31}$$

We could find their communicative relations

$$[J_0, J_+] = J_+, \quad [J_0, J_-] = -J_- \tag{32}$$

and

$$\begin{aligned}
[J_+, J_-] \;=\; & \prod_{i=0}^{l_2-1}\left(\frac{H}{\hbar(\omega_1+\omega_2)} + 2l_2J_0 + (2i-\frac{1}{2})\right)\cdot\left(\frac{H}{\hbar(\omega_1+\omega_2)} + 2l_2J_0 + (2i-\frac{3}{2})\right)\cdot \\
& \prod_{j=0}^{l_1-1}\left(\frac{H}{\hbar(\omega_1+\omega_2)} - 2l_1J_0 - (2j-1) - \frac{1}{2}\sqrt{\frac{4m\kappa}{\hbar^2}+1}\right)\cdot \\
& \left(\frac{H}{\hbar(\omega_1+\omega_2)} - 2l_1J_0 - (2j-1) + \frac{1}{2}\sqrt{\frac{4m\kappa}{\hbar^2}+1}\right) \\
& -\prod_{i=0}^{l_2-1}\left(\frac{H}{\hbar(\omega_1+\omega_2)} + 2l_2J_0 - (2i-\frac{3}{2})\right)\cdot\left(\frac{H}{\hbar(\omega_1+\omega_2)} + 2l_2J_0 - (2i-\frac{1}{2})\right)\cdot \\
& \prod_{j=0}^{l_1-1}\left(\frac{H}{\hbar(\omega_1+\omega_2)} - 2l_1J_0 + (2j-1) - \frac{1}{2}\sqrt{\frac{4m\kappa}{\hbar^2}+1}\right)\cdot \\
& \left(\frac{H}{\hbar(\omega_1+\omega_2)} - 2l_1J_0 + (2j-1) + \frac{1}{2}\sqrt{\frac{4m\kappa}{\hbar^2}+1}\right)
\end{aligned} \tag{33}$$

which the maximal order of J_0 in $[J_+, J_-]$ is $2(l_1+l_2)-1$ corresponding to the polynomial algebras with $2(l_1+l_2)-1$ order. We can solve their Casimir operator

$$\begin{aligned}
C \;=\; & \prod_{i=0}^{l_2-1}\left(\left(\frac{H}{\hbar(\omega_1+\omega_2)}\right)^2 + 2\frac{H(2i-1)}{\hbar(\omega_1+\omega_2)} + 4i(i-1) + \frac{3}{4}\right)\cdot \\
& \prod_{j=0}^{l_1-1}\left(\left(\frac{H}{\hbar(\omega_1+\omega_2)}\right)^2 - \frac{2H(2j-1)}{\hbar(\omega_1+\omega_2)} + 4j(j-1) + \frac{3}{4} - \frac{m\kappa}{\hbar^2}\right) \\
& +\prod_{i=0}^{l_2-1}\left(\left(\frac{H}{\hbar(\omega_1+\omega_2)}\right)^2 - \frac{2H(2i-1)}{\hbar(\omega_1+\omega_2)} + 4i(i-1) + \frac{3}{4}\right)\cdot \\
& \prod_{j=0}^{l_1-1}\left(\left(\frac{H}{\hbar(\omega_1+\omega_2)}\right)^2 + \frac{2H(2j-1)}{\hbar(\omega_1+\omega_2)} + 4j(j-1) + \frac{3}{4} - \frac{m\kappa}{\hbar^2}\right)
\end{aligned} \tag{34}$$

and energy level

$$E_{(1)i,j;n} = 2\frac{\omega_1\omega_2}{\omega_0}n - \hbar\omega_1(2i-\frac{1}{2}) - \hbar\omega_2(2j-1) - \frac{\omega_2}{2}\sqrt{4m\kappa+\hbar^2}; \tag{35a}$$

$$E_{(2)i,j;n} = 2\frac{\omega_1\omega_2}{\omega_0}n - \hbar\omega_1(2i-\frac{1}{2}) - \hbar\omega_2(2j-1) + \frac{\omega_2}{2}\sqrt{4m\kappa+\hbar^2}; \tag{35b}$$

$$E_{(3)i,j;n} = 2\frac{\omega_1\omega_2}{\omega_0}n - \hbar\omega_1(2i-\frac{3}{2}) - \hbar\omega_2(2j-1) - \frac{\omega_2}{2}\sqrt{4m\kappa+\hbar^2}; \tag{35c}$$

$$E_{(4)i,j;n} = 2\frac{\omega_1\omega_2}{\omega_0}n - \hbar\omega_1(2i-\frac{3}{2}) - \hbar\omega_2(2j-1) + \frac{\omega_2}{2}\sqrt{4m\kappa+\hbar^2}. \tag{35d}$$

where $-\frac{\hbar^2}{4m} < \kappa < \frac{3\hbar^2}{4m}$, $n = 0, 1, 2, \cdots$ and there are $n+1$ degenerate eigenstates for each energy level $E_{(s)nij}$, $s = 1, 2, 3, 4$, $i = 0, \cdots, l_2-1$, $j = 0, \cdots, l_1-1$. When $l_1 : l_2 = 1 : 1$, it could be viewed as the Smorodinsky-Winternitz potential; when $l_1 : l_2 = 1 : 2$, it could be viewed as the Holt potential.

IV. DISCUSSION

In this paper, we solve arbitrary integer ratio, $l_1 : l_2$, between two frequencies of 2-dimensional harmonic oscillator. The deformed oscillators could be solved by polynomial algebras. Meanwhile, oscillators with arbitrary integer ratio

frequencies are also real physical model. Actually, with ladder operators, the physical model with equal energy interval can be solved by polynomial algebras. With this practice of 2-dimensional system, we could try to solve 3-dimensional system with expanding su(3) or so(4) to their non-linear form.

Acknowledge

We thank Bo Fu for his helpful discussion and checking the manuscript carefully. This work is supported in part by NSF of China (Grants No. 10975075), Program for New Century Ex- cellent Talents in University, and the Project-sponsored 5 by SRF for ROCS, SEM.

[1] P. W. Higgs, J. Phys. A **12**, 309 (1979).
[2] Leemon, J. Phys. A **12**, 489 (1979).
[3] R. Floreanini, L. Lapointe, and L. Vinet, Phys. Lett. B **389**, 327 (1996).
[4] V. P. Karassiov and A. B. Klimov, Phys. Lett. A **189**, 43 (1994).
[5] A. S. Fokas and P. A. Lagerstrom, J. Math. Anal. Appl. **74**, 325 (1980).
[6] P. Winternitz, Ya. A. Smorodinsky, M. Uhlir, and I. Fris, Yad. Fiz. **4**, 625 (1966)[Sov. J. Nucl. Phys. **4**, 444(1966)].
[7] C. R. Holt, J. Math. Phys. **23**, 1037 (1983).
[8] Dennis Bonatsos, C.Daskaloyannis, and K.Kokkotas, Physical Review A **50**,5, 3700 (1994).
[9] D. Ruan, in *Frontiers in Quantum Mechanics*, edited by J. Y. Zeng, S. Y. Pei, and G. L. Long (Beijing: Beijing University, 2001), p. 344.
[10] D. M. Fradkin, Am. J. Phys. **33**, 207 (1965).